USE OF THE DISCHARGE-WEIGHTED AVERAGE VELOCITY IN STUDIES OF THE FRICTIONAL ENERGY LOSS OF STREAMFLOW

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ABSTRACT

The discharge-weighted average velocity of streamflow is proposed as an alternative to the more commonly used area-weighted average for uses where frictional energy losses are evaluated. Using experimental data from Mink Brook, NH, and data published by Marchand *et al*. (USGS Open File Report 84–733, 1984), the theoretical framework developed here indicates that, through the use of the area-weighted average velocity, Manning's *n* overestimates frictional effects by 2–156 per cent in certain common situations. Variance in the vertical axis, from a sample of various streams in Colorado, creates an additional overestimation of 6–25 per cent. A sample velocity distribution for a smooth walled, straight, trapezoidal channel creates an overestimation of 24 per cent. Similar overestimation may partially explain the roughness multiplier applied to the Nikuradse *K* in gravel-bed streams where particle size approaches or exceeds mean depth. A generalized resistance equation is proposed that may assist in evaluating the range of conditions found in the natural world. © 1997 by John Wiley & Sons, Ltd.

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INTRODUCTION

The work of Chézy, Darcy, Nikuradse and others, beginning in the 18th century, has led to the streamflow resistance equations in use today. The Manning equation, perhaps the most widely used of these formulas, typifies this entire family. Although empirically derived, it is closely related to the theoretically based Darcy–Weisbach equation, itself mathematically similar to a broad range of equations.

The Manning equation was developed for conditions of uniform flow and small particle sizes. While some legitimate criticism has been levelled against the usefulness of Manning-like equations for situations of non-uniform flow, or situations where form drag is a major component of bed friction, researchers have long applied these sorts of equations to a wide variety of streamflow conditions in an effort to develop a theoretical foundation applicable to the breadth of morphologies found in the natural world. Empirically derived photographic handbooks or tables, applicable to many channel conditions, have provided values of *n* to field researchers for almost 40 years (Barnes, 1967; Cowan, 1956; Chow, 1959). More recently, researchers have applied roughness equations based on particle size to reaches with particles of comparable size to the mean depth, conditions of significant form drag and non-uniform flow (Griffiths, 1981; Bathurst, 1977, 1985).

Existing resistance equations were originally based upon studies of manufactured pipes and canals. Historically, they have been used to estimate discharge, often of public works and plumbing projects, and also of flood events. These uses are still common, but they have been joined in recent decades by a new group of research efforts aimed at untangling the frictional effects present in streamflow. Such studies are more concerned with energy loss than discharge, which is often a known factor. Existing resistance equations have been put to this task, a task for which they are theoretically insufficient. Problems arise when applying this theoretical base to conditions manifesting significant variance (Griffiths, 1981). Actually, some degree of velocity variation is evident in all conditions, leading to error when existing methods are used to evaluate frictional effects. The use of the discharge-weighted average velocity in certain research situations is a promising technique to begin to correct such error. In fact, the assumptions inherent in existing resistance

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equations tend to break down in many naturally occurring situations. More generalized theories of resistance may be called for.

CONCEPTUAL FRAMEWORK

The area-weighted average velocity has long been the standard in hydrologic research, primarily because it is a component of the mass-balance equation:

$$Q = A\bar{v}_{\mathrm{area}}$$

where Q is the discharge of the stream, A is the total cross-sectional area, and $\bar{v}_{\mathrm{area}}$ is derived in this fashion:

$$\bar{v}_{\mathrm{area}} = \frac{\int_A v\,\mathrm{d}A}{A}$$

Here dA is the differential of cross-sectional area and v is the velocity of flow through the differential.

$\bar{v}_{\mathrm{area}}$ has been used almost exclusively whenever some value for an average velocity is needed. For certain situations, I propose an alternative – the discharge-weighted average velocity ($\bar{v}_Q$), which is derived in this manner:

$$\bar{v}_Q = \frac{\int_A v^2\,\mathrm{d}A}{Q} = \frac{\int_A v\,\mathrm{d}Q}{Q}$$

where dQ is the differential of discharge passing through the cross-section.

Conceptually, the important difference is that $\bar{v}_{\mathrm{area}}$ represents velocity through a unit of area; $\bar{v}_Q$ represents the velocity of a volume of water. $\bar{v}_Q$ has been used in evaluating sediment concentrations, and has long been an essential component of the average momentum equation (Guy and Simons, 1964; Henderson, 1966). Total fluid momentum has often been described in these terms:

$$M = \beta\bar{v}_{\mathrm{area}}Q\rho$$

where M is the momentum, υ is the density of water, and β is the Coriolis beta coefficient, which is used to correct the $\bar{v}_{\mathrm{area}}$ for variation in velocity. β is derived in this fashion (Idelchik, 1986; Henderson, 1966):

$$\beta = \frac{1}{A}\int_A \left(\frac{v}{\bar{v}_{\mathrm{area}}}\right)^2 \mathrm{dA}$$

which can be rearranged thus:

$$\beta = \frac{1}{\bar{v}_{\mathrm{area}}{}^2 A}\int_A v^2\,\mathrm{d}A = \frac{\int_A v\,\mathrm{d}Q}{\bar{v}_{\mathrm{area}}Q} = \frac{\bar{v}_Q}{\bar{v}_{\mathrm{area}}}$$

When this is reinserted into the momentum equation, the momentum of streamflow becomes:

$$M = \bar{v}_Q Q\rho$$

Since fluid momentum is simply the product of fluid velocity and mass, it becomes apparent that $\bar{v}_Q$ is indeed the average velocity of streamflow (Richards, 1982).

$\bar{v}_Q$ will be equal to $\bar{v}_{\mathrm{area}}$ for hypothetical conditions of uniform velocity (no variation), but since bed surface friction will always introduce some variation, $\bar{v}_Q$ will be larger than $\bar{v}_{\mathrm{area}}$ for any given cross-section. The

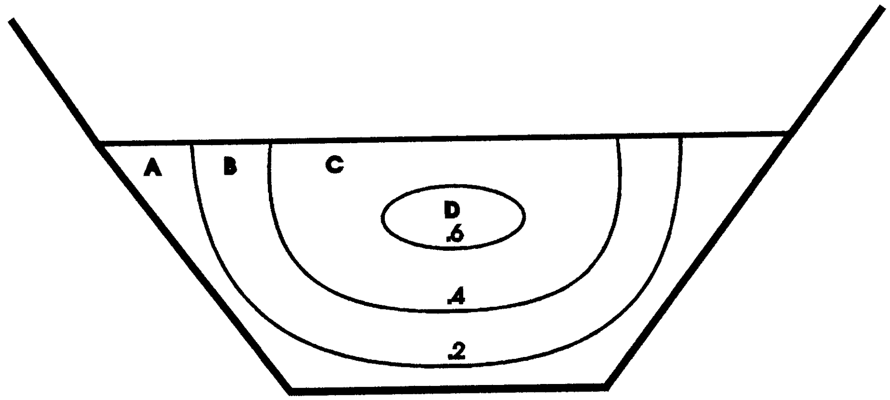

Figure 1. A cross-section of a hypothetical straight, smooth-walled, trapezoidal channel. After Richards (1982)

magnitude of the difference between the two depends on the degree of variance in velocity. Reaches with a large variance, stretches with meanders, obstructions, large particles, pools, vegetation, etc., will manifest a greater difference between the two averages.

A brief example demonstrates the conceptual and pragmatic differences between $\bar{v}_Q$ and $\bar{v}_{\text{area}}$. Take a hypothetical cross-section across a smooth-walled, straight, trapezoidal channel (Figure 1). For computation purposes, the flow is divided into four subsections, bounded by isovels. The average velocity of the discharge passing through the subsection is assumed to be the average of the two bounding values, except for the central one, which was assigned a velocity of 0·65. Table I summarizes the results. In the slowest subsection, subsection A, 7 per cent of the discharge is moving through 23 per cent of the area. In the fastest, 8 per cent of the discharge is moving through 4 per cent of the area. Computing $\bar{v}_{\text{area}}$ allows one to find the total discharge by means of the mass-balance equation, but in terms of the average velocity of the discharge, it overweights slower-moving fractions of the cross-section and underweights fast-moving ones. $\bar{v}_Q/\bar{v}_{\text{area}}$ for the cross-section is 1·24.

Table I. Example of the derivation of the area-weighted and discharge-weighted average velocities for the hypothetical cross section in Figure 1

Subsection	*A*	v	*Q*	%*A*	%*Q*
A	58	0·1	5·8	23	7
B	90	0·3	27·0	36	32
C	91	0·5	45·5	36	53
D	11	0·65	7·2	4	8
Total	250		85·5	100	100

$\bar{v}_Q$	0·42
$\bar{v}_{\text{area}}$	0·34
$\bar{v}_Q/\bar{v}_{\text{area}}$	1·24

METHODS AND RESULTS

To examine the degree to which $\bar{v}_Q$ and $\bar{v}_{\text{area}}$ differ in real situations, velocity at a number of cross-sections on Mink Brook in Hanover, New Hampshire, was measured using a current meter and the corresponding averages computed (Table II). This study was not intended as a comprehensive investigation into channel conditions, but only as a demonstration of the magnitude of the differences between the $\bar{v}_Q$ and $\bar{v}_{\text{area}}$ in a number of naturally occurring channels bearing common morphologies.

Table II. $\bar{v}_Q \bar{v}_{area}$ computed for 17 cross-sections on Mink Brook, Hanover, NH. All velocities in metres per second

Site	$\bar{v}_{area}$	$\bar{v}_Q$	$\bar{v}_Q/\bar{v}_{area}$
Highly obstructed no. 1	0·82	1·73	2·11
Highly obstructed no. 2	1·08	1·62	1·49
Highly obstructed no. 3	1·15	2·57	2·23
Moderately obstructed no. 1	1·46	2·41	1·65
Moderately obstructed no. 2	0·81	1·29	1·59
Moderately obstructed no. 3	0·64	1·06	1·65
Meanders no. 1	0·34	0·60	1·79
Meanders no. 2	0·31	0·53	1·69
Meanders no. 3	0·30	0·40	1·36
Pool no. 1	0·14	0·37	2·58
Pool no. 2	0·36	0·49	1·36
Pool no. 3	0·34	0·85	2·52
Riffle no. 1	2·51	2·55	1·02
Riffle no. 2	2·45	2·76	1·13
Unclassified no. 1	1·25	2·23	1·78
Unclassified no. 2	0·86	1·07	1·24
Unclassified no. 3	0·68	0·80	1·16

Cross-sections are categorized by channel morphology: meanders, pools (in roughly straight pool–riffle sequences), moderately obstructed flows (largest particles approximately equal to stream depth), heavily obstructed flows (many boulders higher than water level, water channelized into jets, large calm eddies behind), riffles, and a few unclassified reaches. The meandering reaches also displayed pool–riffle features. Flow at the time of the study can be characterized as medium–low flow. Methods used correspond to standard methods for measuring velocity and discharge with current meters, using the six-tenths method of averaging velocity in the vertical (Buchanan and Somers, 1969). Therefore, the Mink Brook data represent velocity variance along a horizontal axis.

The magnitude and range of $\bar{v}_Q/\bar{v}_{area}$ in this small example (2–158 per cent difference between the two variables) underscores the importance of using an averaging technique appropriate to research needs. While certain reaches of Mink Brook represent obstructed, non-uniform flow, other reaches manifesting high $\bar{v}_Q/\bar{v}_{area}$ values are unobstructed and locally uniform. Meanders and pools showed ratios equivalent to those found in the most obstructed reaches in the study. Riffles, where particle sizes approach mean depth, had the least variation, although the scale of variance and turbulence was probably too small to differentiate.

Velocity also varies along the vertical axis, generally as a log-linear relationship to particle size (Wiberg and Smith, 1991). A computation of $\bar{v}_Q/\bar{v}_{area}$ along vertical axes was made for 11 mountainous and lowland streams in Colorado, at three different flows, using data from Marchand *et al*. (1984). $\bar{v}_Q/\bar{v}_{area}$ varied from 1·06 to 1·25 along the vertical axis in these streams.

DISCUSSION

The Manning equation, when using SI units, is as follows:

$$\bar{v}_{area} = \frac{R^x S^{1/2}}{n}$$

where R is the hydraulic radius, defined as the area/wetted perimeter, S is the slope, x is an empirically derived exponent typically between 0·65 and 0·84, and n is the roughness coefficient, which is intended to represent the frictional characteristics of the streambed (Idelchik, 1986).

The Darcy–Weisbach equation is:

$$f = \frac{8gRS}{\bar{v}^2_{\text{area}}}$$

where g is the acceleration due to gravity. Inspection will show that, except that the exponent $x \neq 0{\cdot}5$, they are essentially identical. Manning's n and the Darcy–Weisbach f will be used somewhat interchangeably in the following discussion.

Friction can be defined as the loss of potential and kinetic energy over a stream reach, which Leopold *et al.* (1960) differentiated into skin friction, internal distortion, and spill resistance. Many studies of the micromechanics of streamflow involve intricate accountings of variations in water surface height. For the examples specifically addressed in this study, field estimates of n and resistance in gravel-bed streams, the loss in potential energy is generally averaged by measuring slope over a reach.

In these equations, kinetic energy is represented by average velocity. Here lies the importance of $\bar{v}_Q$ – the mass that, due to its position and velocity, contains energy, is a parcel of water (dQ). Friction removes energy (i.e. velocity), also from parcels of water. Therefore, if we wish to use these equations solely as a measure of frictional effects on water velocity (and not to estimate discharge), we must incorporate the average velocity of parcels of water ($\bar{v}_Q$). The use of $\bar{v}_{area}$ for the purpose of describing frictional effects introduces an inflation of n proportional to $\bar{v}_Q/\bar{v}_{area}$. This overestimation is solely due to velocity variance within the cross-section, and for the most part does not represent any sort of frictional effect, including skin friction, internal deformation, and form drag. The data from Mink Brook demonstrate that in common, naturally occurring situations, this overestimation may exceed 100 per cent. Whenever there exists some variance in velocity (the case in all channels), some non-frictional inflation of n also exists, however small.

In other words, Manning's n as it now stands, is a function not only of friction but also of the degree of variance in velocity in the cross-section, which has little to do with the total energy (potential and kinetic) of streamflow. This situation is acceptable if one's goal is to produce $\bar{v}_{area}$ and thus estimate discharge, but it is not satisfactory if one's goal is to judge frictional effects alone. Two values of n, or f, may be called for.

The kinetic energy of flow is derived in this fashion (Richards, 1982):

$$E_k = \frac{1}{2}\alpha \bar{v}^2_{\text{area}} Q\rho$$

where E_k is the kinetic energy of flow, υ is the density of water, and α is the Coriolis alpha coefficient, which serves to correct $\bar{v}_{area}$ for variation in velocity. α is derived as follows (Idelchik, 1986):

$$\alpha = \frac{1}{A}\int_A \left(\frac{v}{\bar{v}_{\text{area}}}\right)^3 \mathrm{d}A$$

This can be rewritten as:

$$\alpha = \frac{1}{\bar{v}^3_{\text{area}} A}\int_A v^3 \mathrm{d}A = \frac{\int_A v^2 \mathrm{d}Q}{\bar{v}^2_{\text{area}} Q}$$

Thus:

$$E_k = \frac{1}{2}\left(\int_A v^2 \mathrm{d}Q\right)\rho$$

And the average kinetic energy is:

$$\bar{E}_k = \frac{1}{2}\frac{\int_A v^2 \mathrm{d}Q}{Q}\rho$$

which is exactly what we would expect. One hopes this example will illustrate the relationship between $\bar{v}_Q$ and energy loss (friction). The continuing use of $\bar{v}_{\mathrm{area}}$ in equations of momentum and energy is puzzling. One suspects that its presence is primarily a quirk of history.

Friction of flow equations based on the Nikuradse roughness value K were extended to open channel flow over 50 years ago. However, in streams where particle sizes were comparable to the mean depth, typically gravel-bed steep mountain streams, a multiplier of roughness length was deemed necessary by a series of empirical experiments, producing the odd situation where the 'effective roughness height' is larger than all but the very largest existing particles (both Clifford *et al*. (1992) and Aguirre-Pe and Fuentes (1990) have excellent, detailed descriptions of research in this area).

A number of studies have addressed this problem. Wake effects, large-scale gravel bars, and particle clustering have all been identified as possible explanations of the roughness multiplier in gravel-bedded streams (Clifford *et al*., 1992; Aguirre-Pe and Fuentes, 1990; Hey, 1988). Bathurst's 1985 study included reaches where $\bar{d}/D_{84}$ attained values as low as 0·43, denoting the presence of a large number of particles protuding from the water's surface (on Mink Brook, this would probably be classified as 'heavily obstructed'). While form drag increases in significance as particle sizes approach mean depth, one must also recognize the inflation of n due simply to the increasing variance in velocity in the cross-section under these conditions. A certain component of the roughness multiplier applied to gravel-bed streams is due to the non-frictional inflation of n inherent in the use of $\bar{v}_{\mathrm{area}}$. The degree of inflation is proportional to $\bar{v}_Q/\bar{v}_{\mathrm{area}}$. This is to say, the use of $\bar{v}_Q$ facilitates proper assessment of the frictional energy loss related to form drag.

The Darcy–Weisbach equation was developed for certain civil engineering purposes, typically involving pipes and manufactured channels. It includes a number of assumptions which may be valid in the situations for which it was designed, and for natural streambeds which resemble manufactured channels (straight, small particles, uniform flow with little variation in velocity) but which become less useful as the equation is applied to many other naturally occurring streambeds (Richards, 1982).

The first assumption is that resistance is a function of total bed area to total volume of water, or in two-dimensional terms, that resistance is a function of the wetted perimeter R (Richards, 1982). This assumption is strong in manufactured situations where each fraction of the bed is playing an identical role in resisting streamflow (e.g. each square centimetre of a concrete channel is experiencing the same shear stress, and applying roughly identical frictive forces to the flow; friction will increase as more concrete is brought into contact with a given amount of water). However, in natural situations, this is often not the case. The streambed contacting the eddy on the inside bend of a sharp meander may be experiencing very little shear stress, while the area of the streambed on the outside curve of the meander may be experiencing much greater than average shear stress, as is evidenced by the differences in particle sizes one will find in each location. We could make the eddy much larger, adding to the wetted perimeter, without increasing friction appreciably. Alternatively, we could remove it entirely. On a smaller scale, where particles are large enough to induce form drag, a portion of the bed behind a large particle may be within that particle's 'eddy', while a portion of the bed immediately adjacent may be experiencing above-average shear stress. The division between streambed convolutions that represent 'particle size', convolutions that represent 'the wetted perimeter', and convolutions that represent 'large-scale bedforms' (e.g. dunes), is often difficult to interpret. Individual particles in gravel-bedded streams are often larger than bedforms in small-particle streams, and measuring the wetted perimeter of a gravel-bed stream is often analogous to measuring the coastline of Britain.

Secondly, as the above demonstrates, the Darcy–Weisbach friction factor was developed to represent the skin friction of different manufactured materials, and the small-particle streambeds that resemble manufactured materials. Currently, it is being used to represent friction from all factors, including form drag and spill resistance (Richards, 1982).

Thirdly, the exponent of the average velocity in the Darcy–Weisbach equation (i.e. 2) is based on the hypothesis that shear stress (and therefore frictional loss) experienced by a portion of the streambed increases with the square of velocity (Richards, 1982). This assumption has been called into question by researchers (Richards, 1982), as evidenced by the range of exponents ($x = 0{\cdot}65$ to $0{\cdot}84$) applied to R in the empirically derived Manning equation. It is interesting to note that the empirically derived range does not include the theoretically derived value. In particular, sources of friction other than skin friction are known not to hold to this assumption (Richards, 1982).

A resistance equation which includes none of these assumptions may be the following:

$$f_Q \bar{v}_Q = S$$

where f_Q is a friction factor, and S is the slope. Intuitively, when $f_Q = 0$ (no friction), and $S>0$, the velocity is infinite, and when $S>0$ and $\bar{v}_Q = 0$, the resistance is infinite. The equation is a simple relation between velocity, friction, and the force of gravity (S) – in this case, f_Q truly represents frictional effects from all sources, including streambed geometry. It is a measure of the frictional resistance experienced by the average unit of discharge. Naturally, it would look nothing like the Darcy–Weisbach f, which is intended as a representation of the skin friction characteristics of channel materials. It may be interesting to measure f_Q in terms of changes in discharge, velocity, slope, sinuosity, etc. In any case, the potential errors inherent to the use of $\bar{v}_{\mathrm{area}}$ in studies of friction and energy loss are to some degree embedded in the equations in which $\bar{v}_{\mathrm{area}}$ is found. Interest in non-manufactured conditions may also predicate the use of alternative equations of resistance. An alternative form for situations where measuring total discharge is the research goal may be:

$$f_{\mathrm{area}} \bar{v}_{\mathrm{area}} = S$$

CONCLUSIONS

Existing resistance equations are designed, and have been historically used, to estimate discharge from manufactured conduits and streambeds which resemble them. To accomplish this end, they produce a value for the area-weighted average velocity, which can then be multiplied by the cross-sectional area to produce an estimate of discharge. In more recent times, researchers have also begun to study energy loss, a use for which the area-weighted average is inappropriate. $\bar{v}_Q$ is proposed as an alternative to the area-weighted average velocity ($\bar{v}_{\mathrm{area}}$) in certain common situations where $\bar{v}_{\mathrm{area}}$ fails to provide what researchers demand of it.

The assumptions inherent to existing resistance equations are not applicable in a great many natural situations. Also, they may not be applicable to studies of frictive energy loss. More generalized resistance equations should provide the initial foundation to a theoretical description of the range of channel conditions in the natural world.

The examples of the use of $\bar{v}_Q$ in this paper are not the only conceivable applications. Use of $\bar{v}_Q$, and more generalized resistance equations, may help explain and ameliorate some of the difficulties experienced by Hey (1979) in analysing pool–riffle conditions, or Dingman (1989) in interpreting velocity distributions. Since dye inherently represents the parcel of water that contains it, understanding $\bar{v}_Q$ may aid the interpretation of tracer results, particularly in obstructed channels or conditions of significant lateral mixing. The use of $\bar{v}_Q$ may also aid studies of sediment transport and other topics involving friction and velocity in stream channels.

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